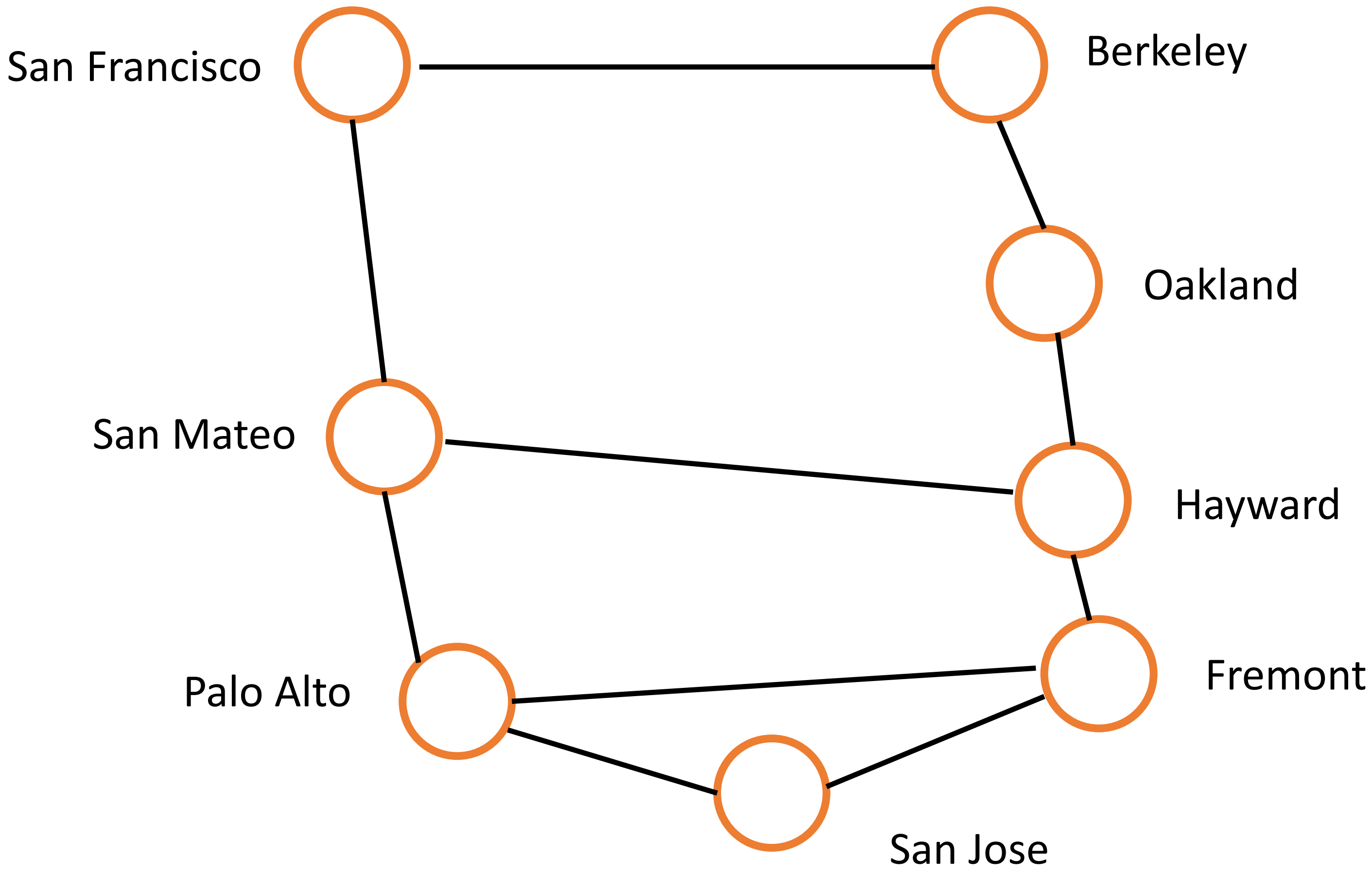
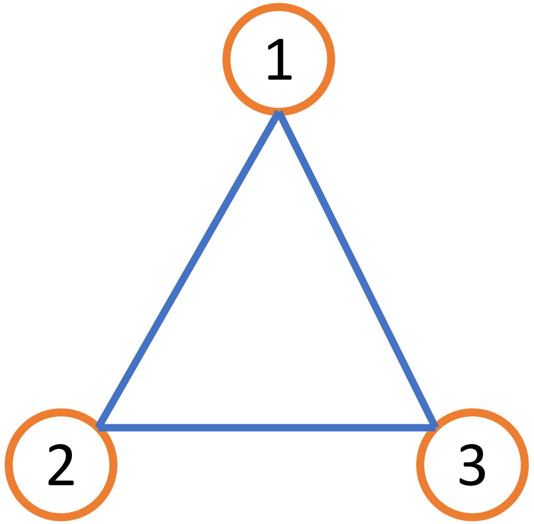
Lecture 6 & 7: Graphs I & II



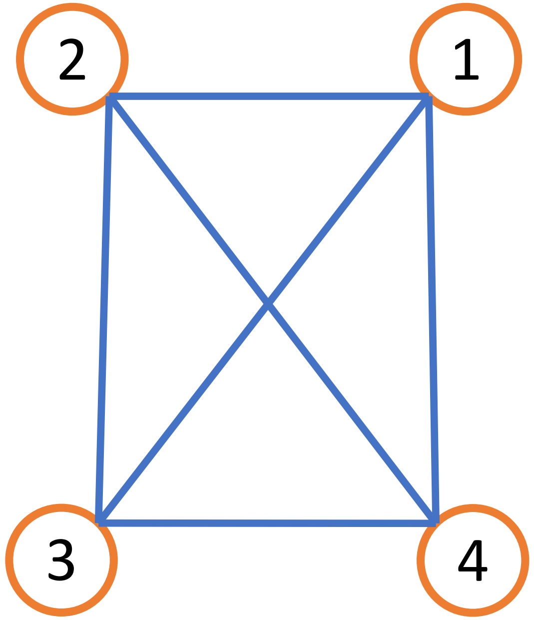
Our Plan

- Basic Notions.
  - Graphs
  - Path / walks / cycles.
- Eulerian Tours
  - Existence
  - Algorithm
- Different kinds of graphs
  - Complete Graph / Trees / Hypercube
- Planar graphs
  - Euler's Formula
  - Five coloring theorem

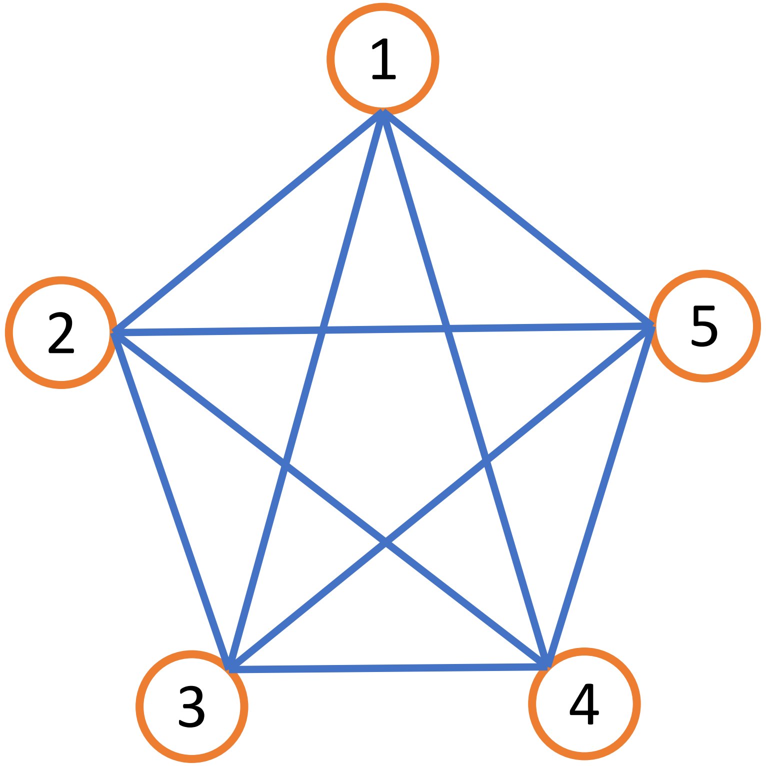
Complete graphs (Cliques)



K_3



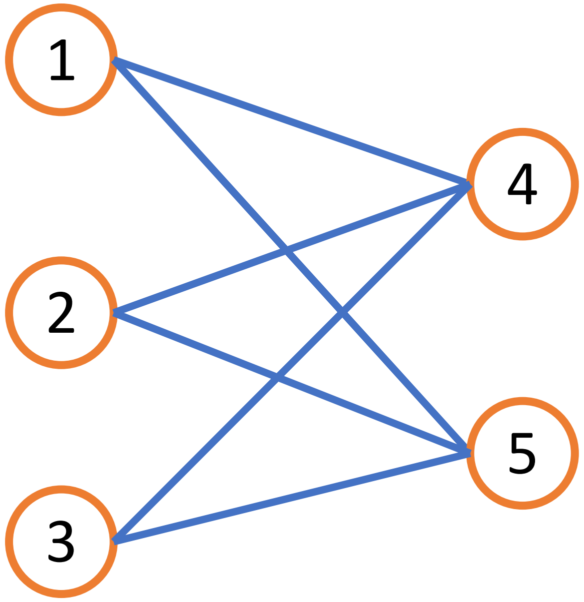
K_4



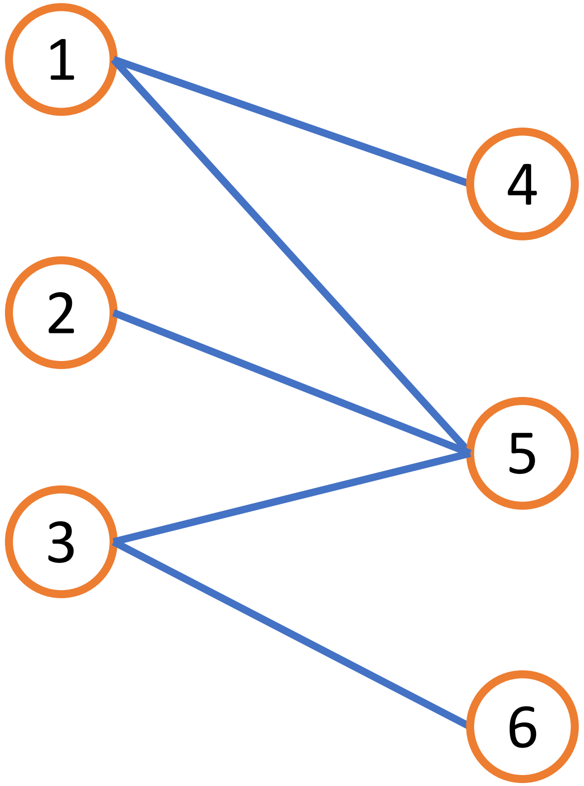
K_5

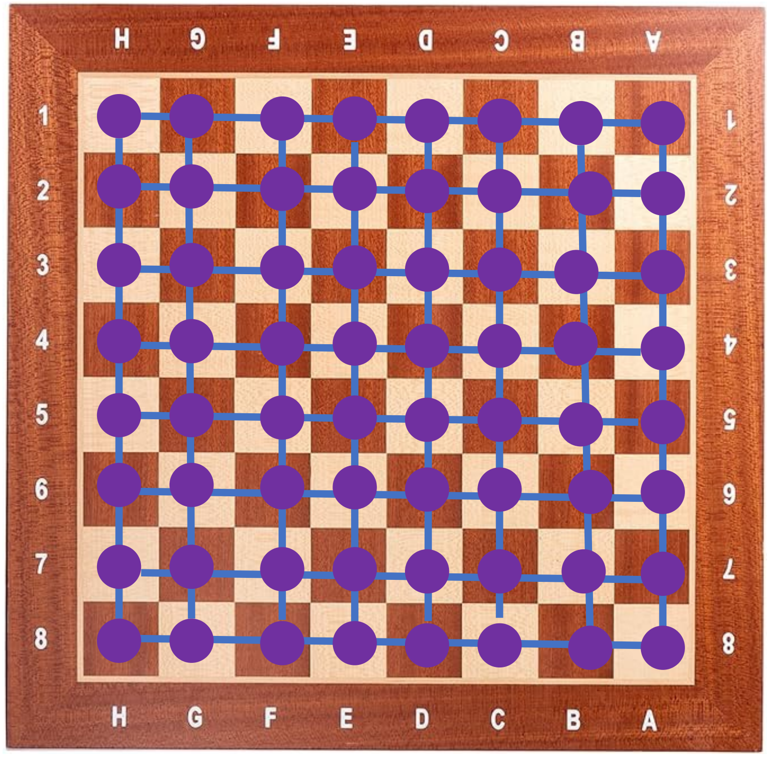
$e = \frac{\mathrm{v}(\mathrm{v}-1)}{2}$ (handshaking lemma)

Bipartite Graphs



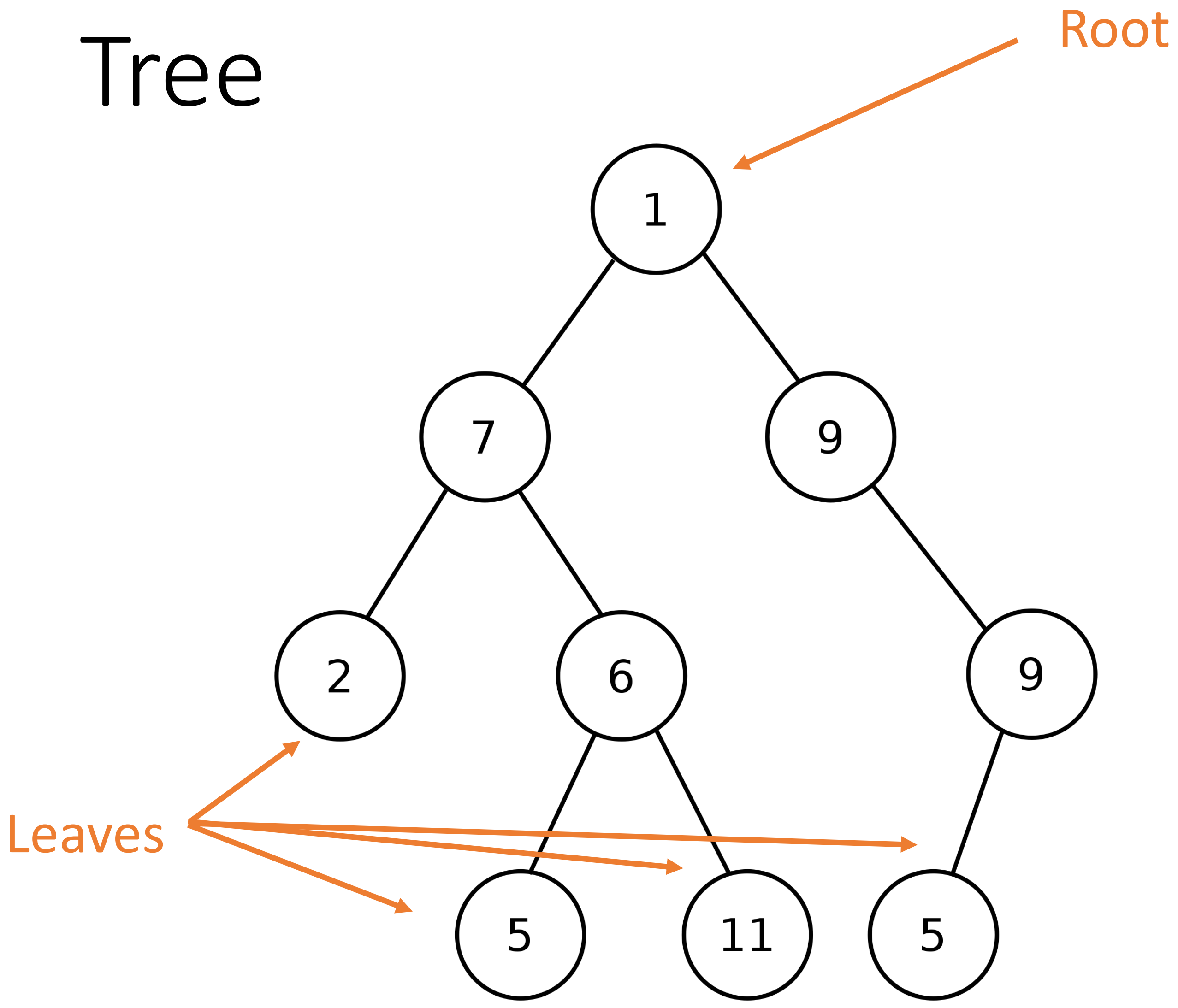
$K_{3,2}$
(bi-clique)



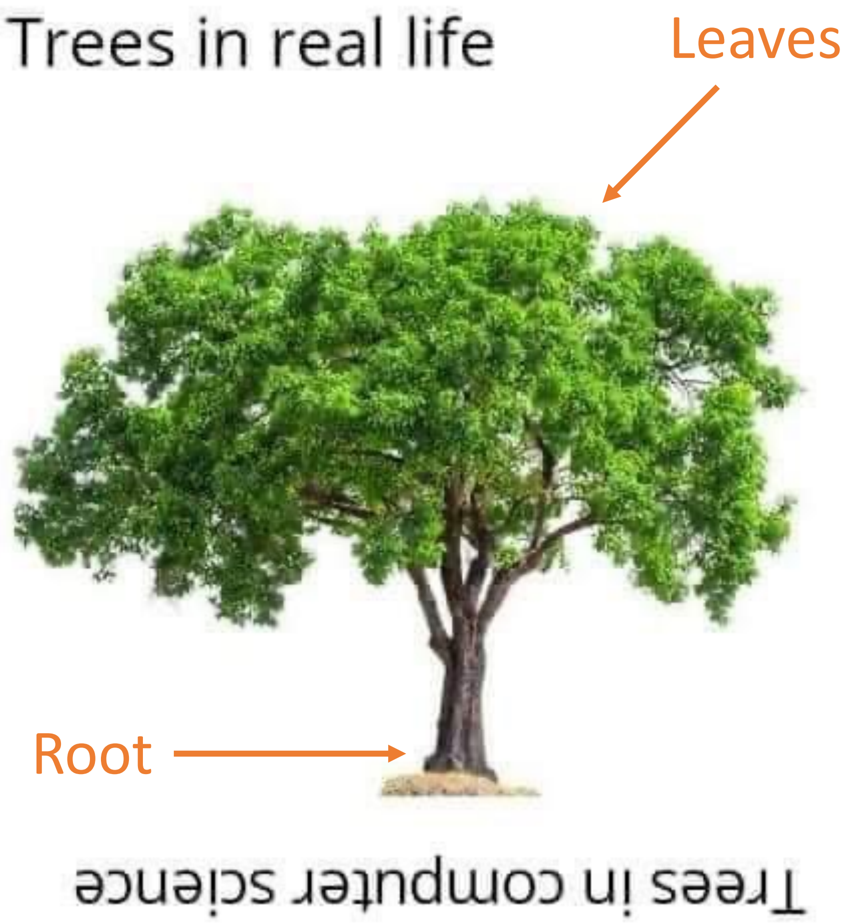


grid

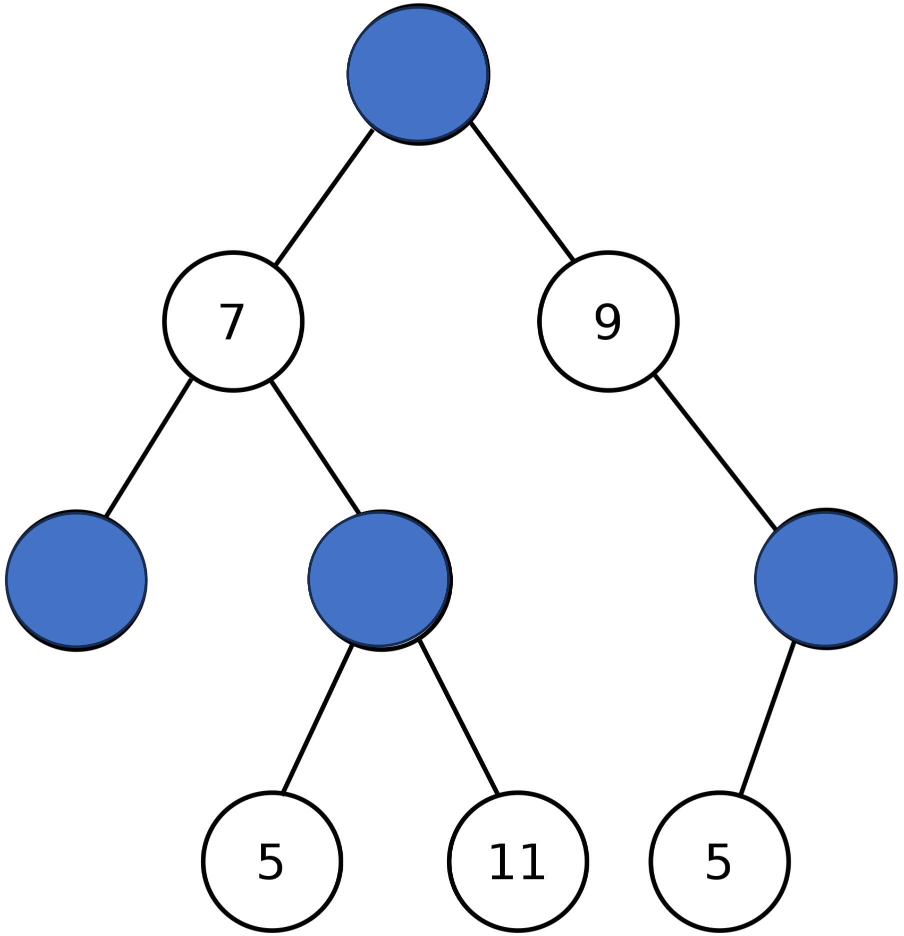
Tree



Connected Acyclic undirected graph



Tree is a bipartite graph



Tree has v – 1 edges

Proof.

Base case: When v = 1, the tree has no edge.

Induction Hypothesis: Suppose this is true for all tree with < v – 1 vertices.

Inductive Step: Take an arbitrary tree with v vertices.

We remove a leaf from it. Now it has v – 1 vertices.

We then add back the leaf, one more edge.

An connected graph with v – 1 edges is a tree

Tree: Connected Acyclic undirected graph

Proof (Attempt).

Base case: When v = 1, the graph with no edge is acyclic.

Induction Hypothesis: Suppose this is true for all tree with < v – 1 vertices.

Inductive Step:

Take an arbitrary connected graph with v vertices and v – 1 edges.

We remove a vertex from it. Now it has v – 1 vertices.

Now how many edges left? Is the graph still connected???

An connected graph with v – 1 edges is a tree

Tree: Connected Acyclic undirected graph

Observation.

There must be a vertex with degree-1 in this graph.

Proof.

Average-degree $= \frac{2(v-1)}{v} < 2.$ (handshaking lemma)

An connected graph with v – 1 edges is a tree

Tree: Connected Acyclic undirected graph

Proof.

Base case: When v = 1, the graph with no edge is acyclic.

Induction Hypothesis: Suppose this is true for all tree with < v – 1 vertices.

Inductive Step:

Take an arbitrary connected graph with v vertices and v – 1 edges.

We remove a degree-1 vertex from it.

Now it has v – 1 vertices and v – 2 edges.

Since the vertex we remove is degree-1, it cannot be on any path/cycle.

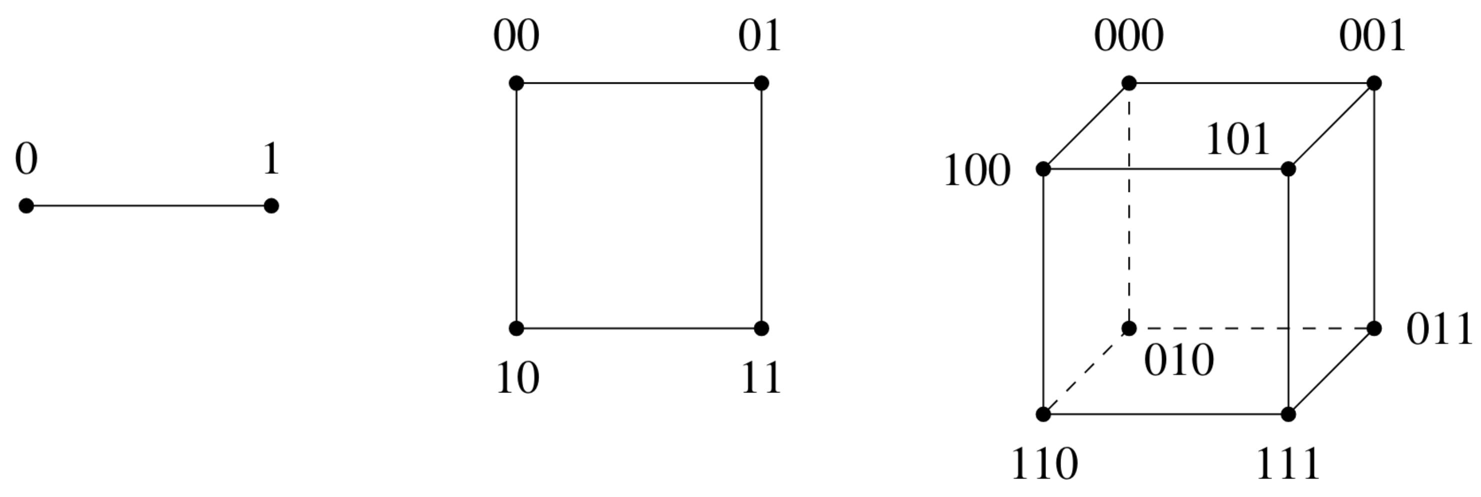
The graph is still connected. By Induction Hypothesis, it is a tree.

After adding the vertex back, it is still connected & acyclic.

Hypercube

Definition-1.

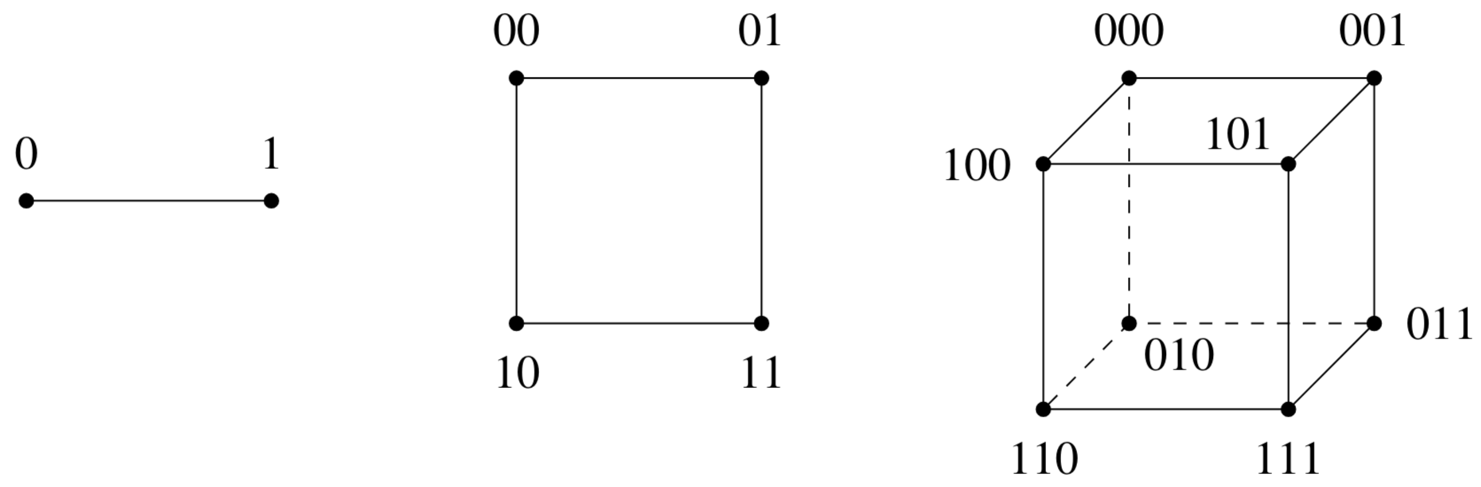
Hypercubes are graphs with vertex set $V = \{0,1\}^n$ (all binary strings) and edge set $E = \{(u, v) |\ u, v \in \{0,1\}^n,\ u, v \text{ only differ in one place}\}$.



Hypercube

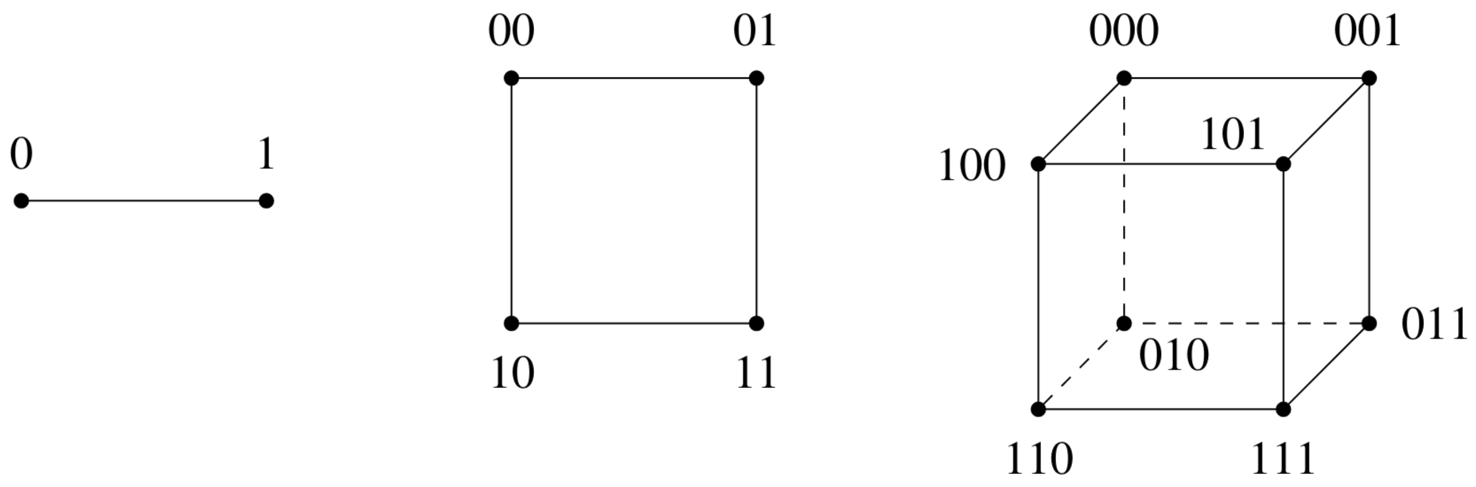
Definition-2.

Hypercubes of dimension n is defined by taking two copies of hypecubes of dimension n – 1 and connect corresponding vertices by edge.



Hypercube

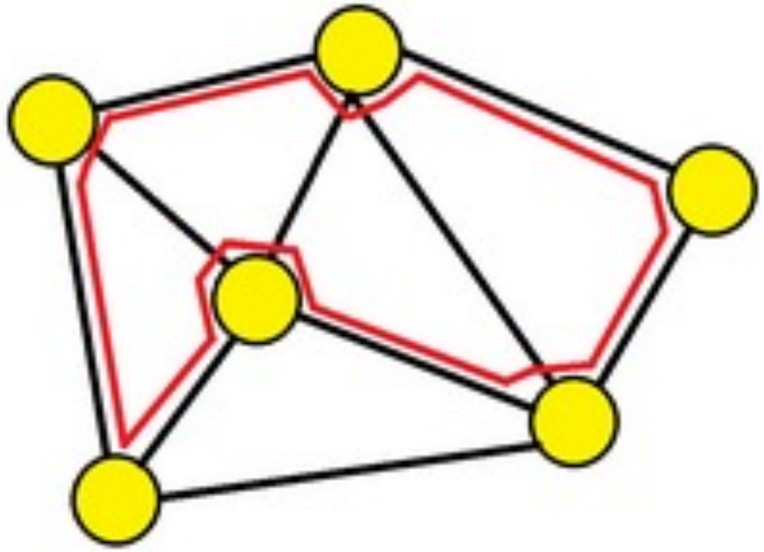
#edge = $n\ 2^{n-1}$ (handshaking lemma).



Hamiltonian Cycle

Theorem.

Hamiltonian Cycle is a cycle that goes through each vertex in the graph exactly once.

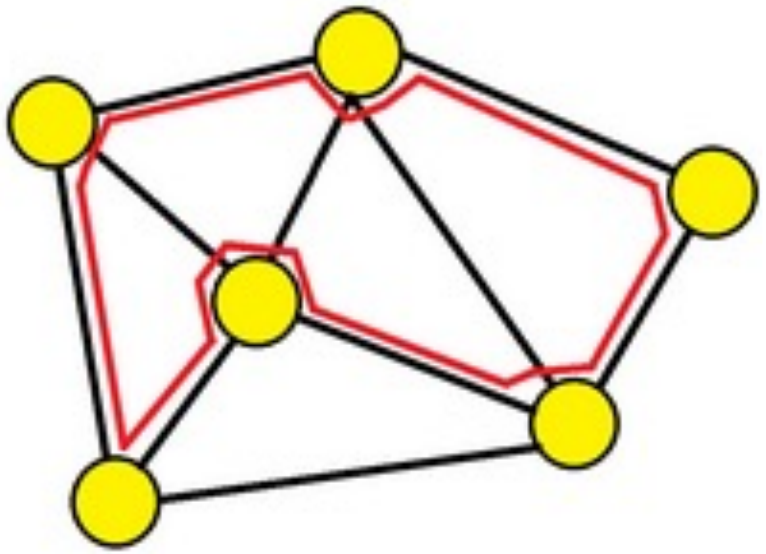


Hamiltonian Cycle

Definition.

Hamiltonian Cycle is a cycle that goes through each vertex in the graph exactly once.

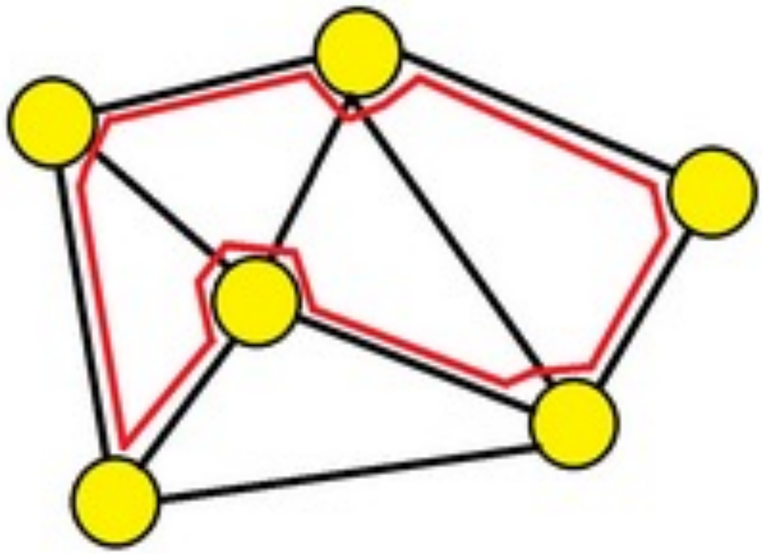
Unlike Eulerian walks, there is no efficient algorithm for finding Hamiltonian Cycles.



Hamiltonian Cycle

Theorem.

If a graph has minimum degree $\geq \frac{n}{2}$, then there is a Hamiltonian Cycle.



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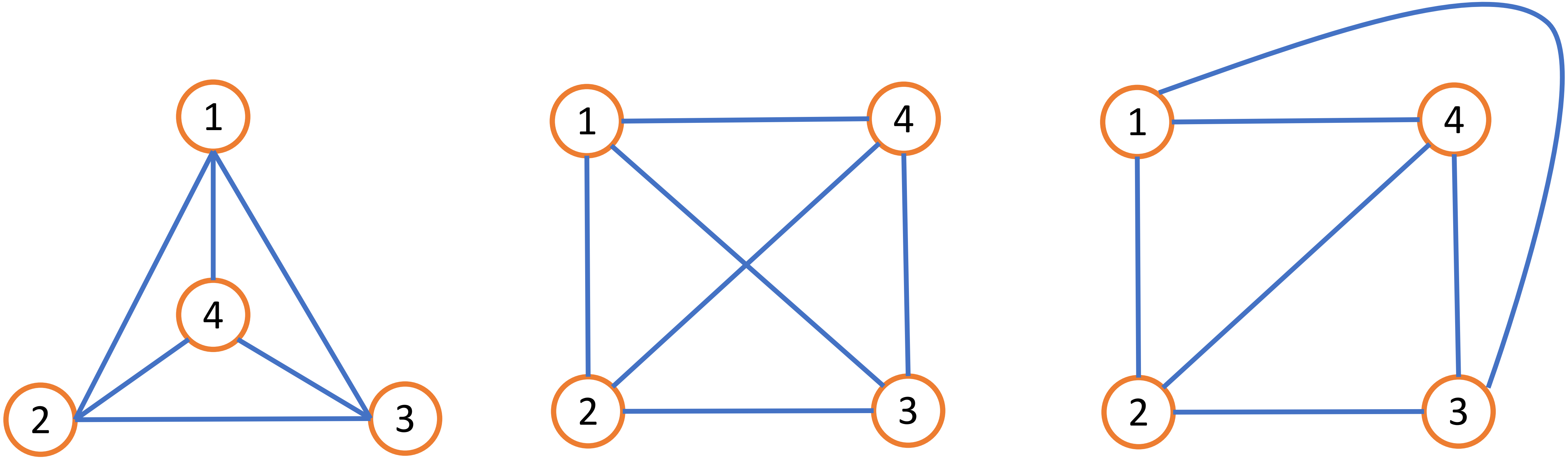
Planar Graph

Observation

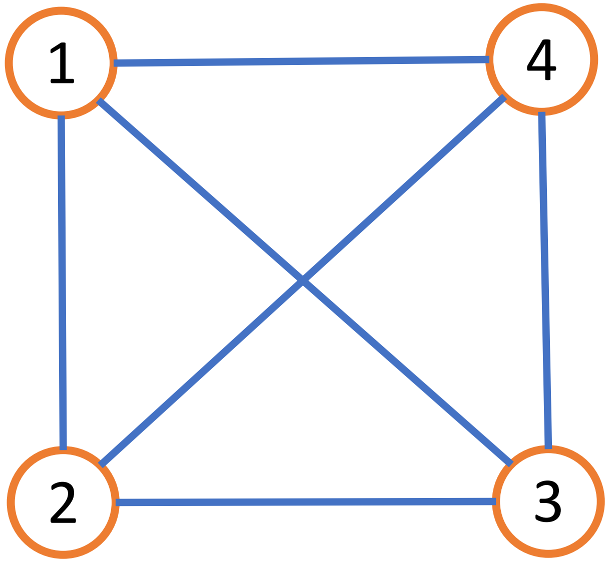
The same graph can be drawn in very different ways.



Planar Graph

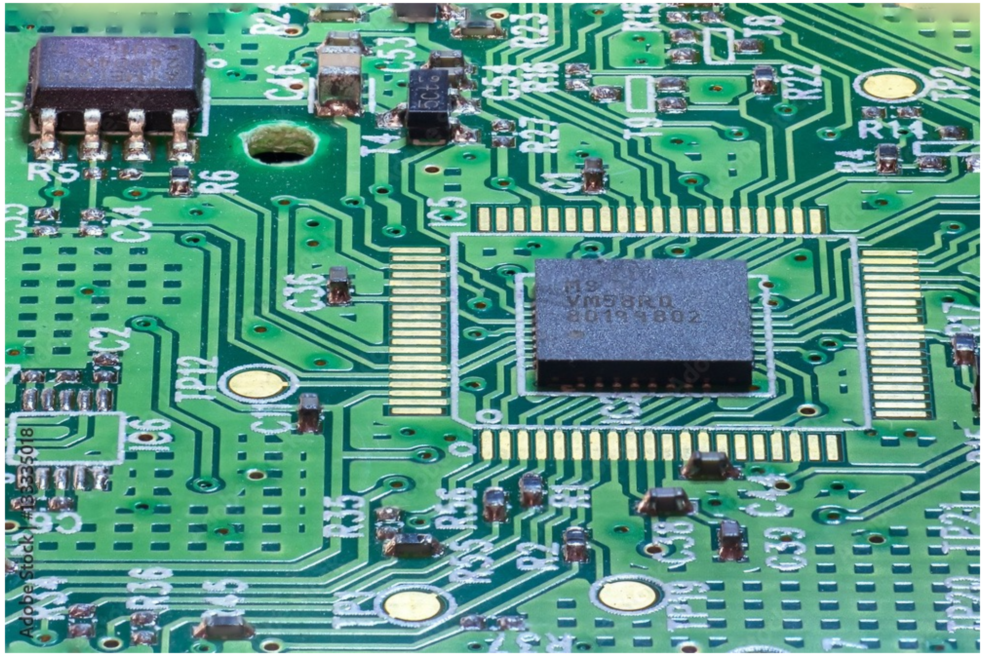
Definition

A planar graph is a graph that can be drawn on a plane without crossing edges.

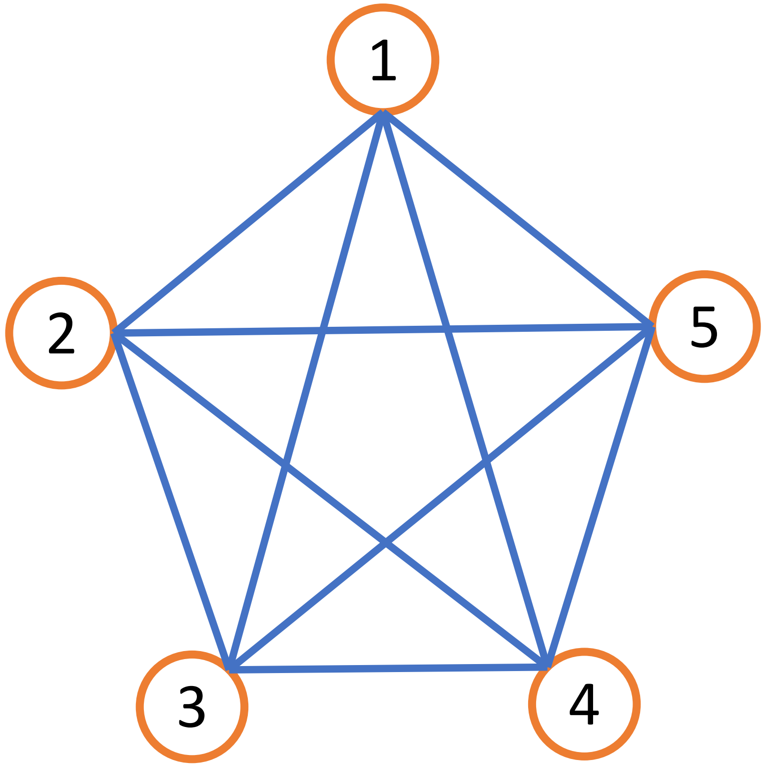


Is this a planar graph?

Why does planar graphs matter?



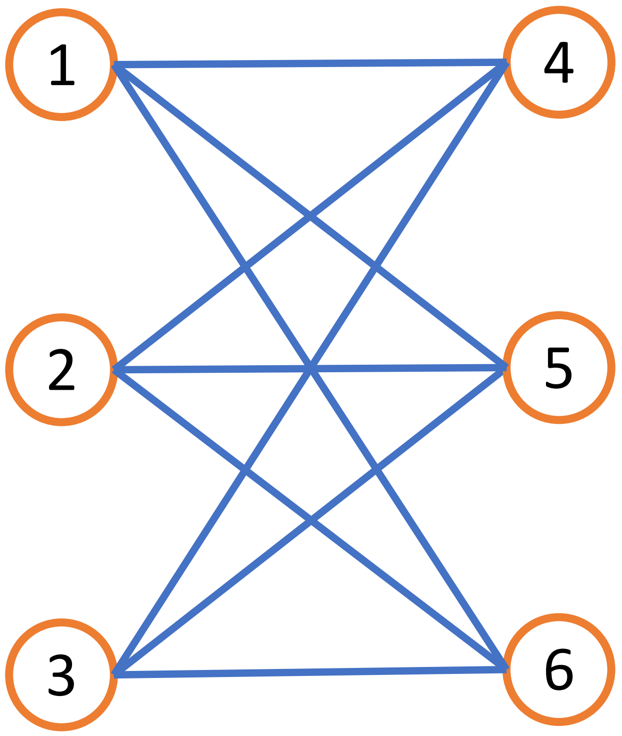
Famous non-planar graphs



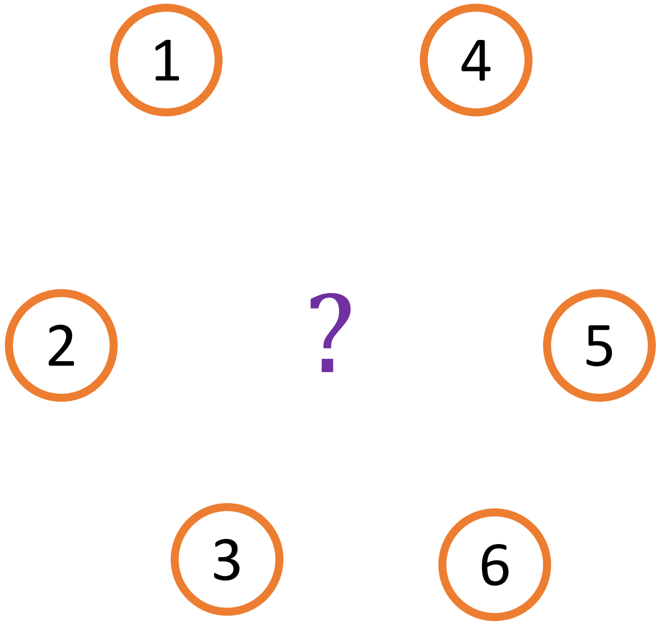
K_5

Magic circle

Famous non-planar graphs



$K_{3,3}$

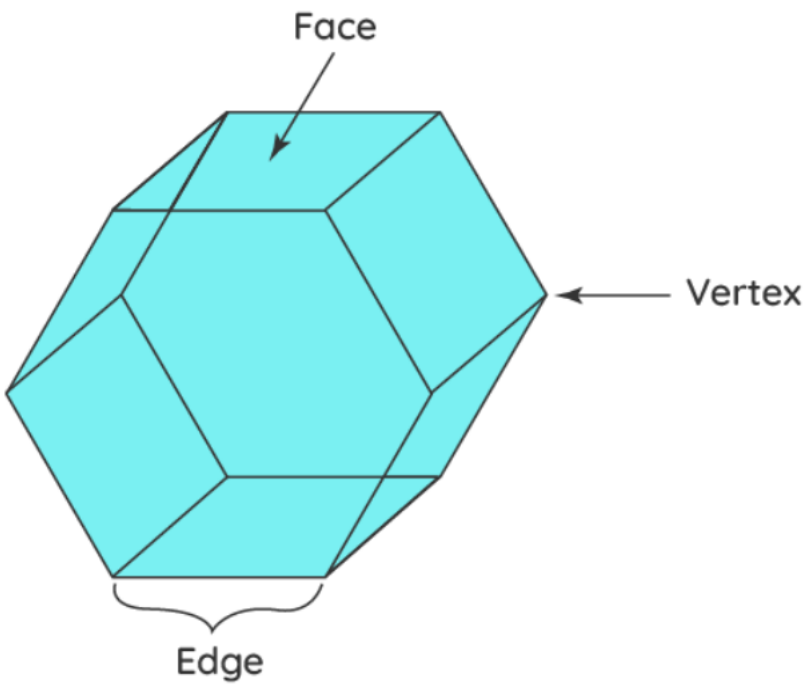


Any graph that contains K_5 or $K_{3,3}$ as a subgraph. /
Any graph with too many edges (e > 3v – 6)
(will prove this later!)

Euler's Formula

Theorem (Since ancient Greeks)

A polyhedral satisfies $v + f - e = 2.$



Euler’s Formula

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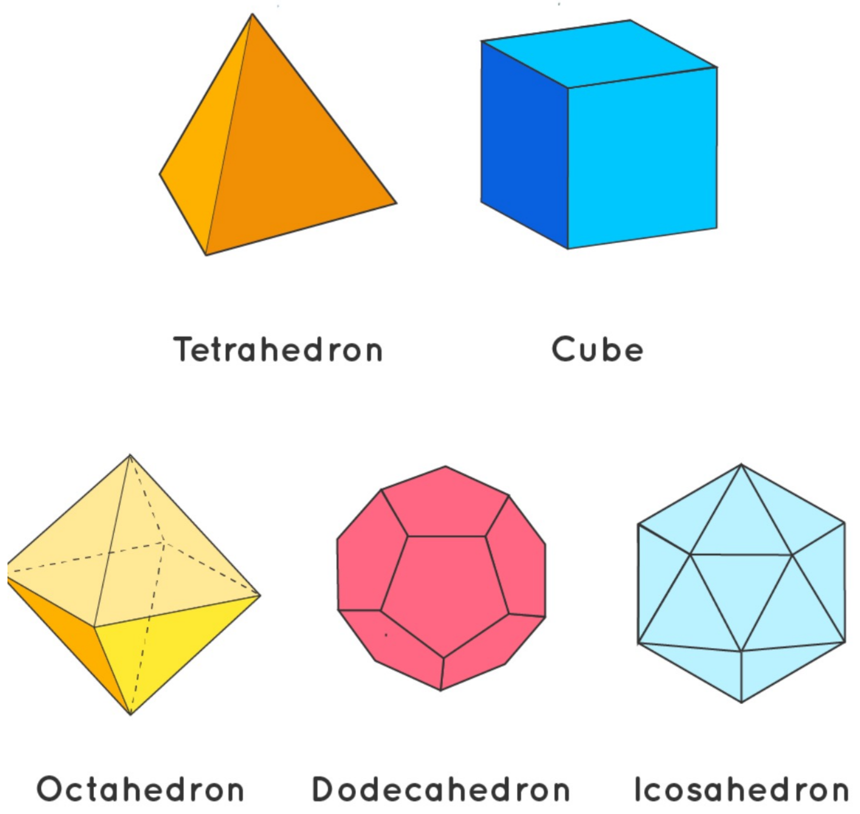


Euler's Formula

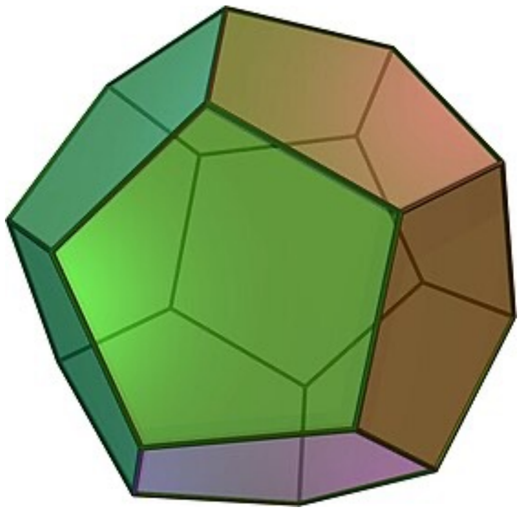
Theorem (Since ancient Greeks)

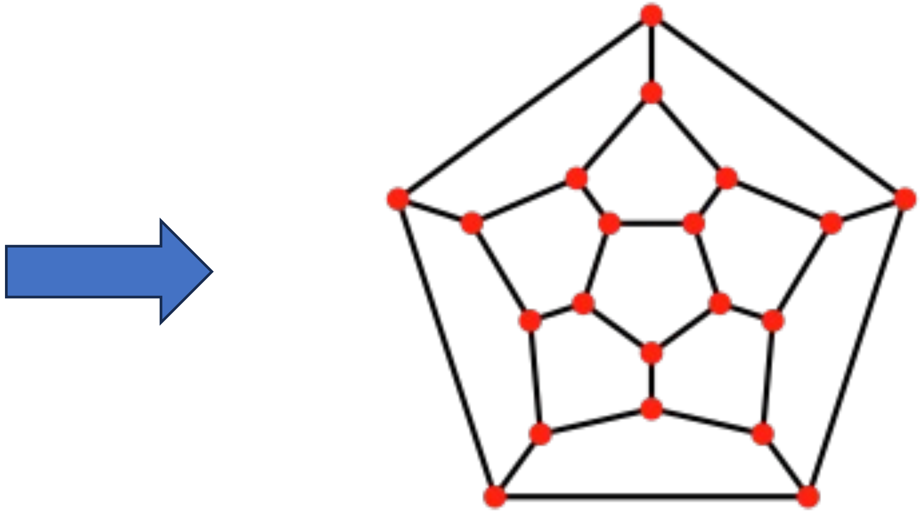
A polyhedral satisfies $v + f - e = 2.$

But ancient Greek don't know how to prove it because they didn't take 70.

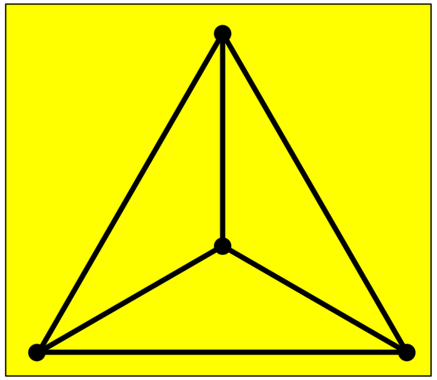
The key is to "strengthen induction hypothesis"

Polyhedrals are planar graphs

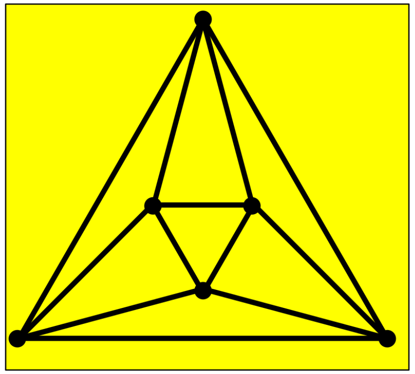




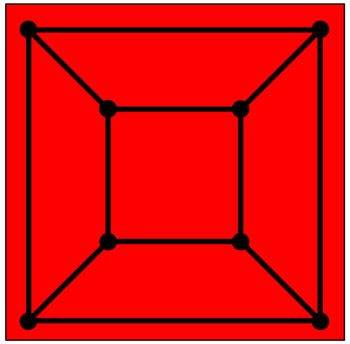
Polyhedrals are planar graphs



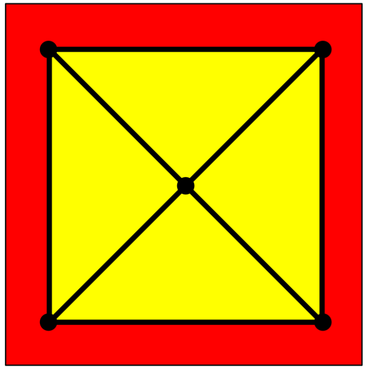
Tetrahedron



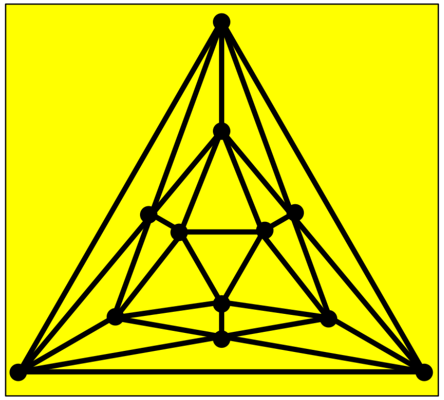
Octahedron



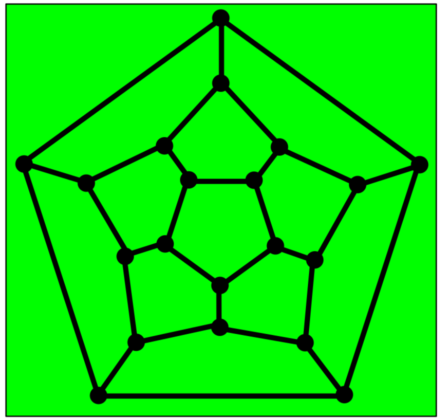
Hexahedron



Square pyramid



Icosahedron

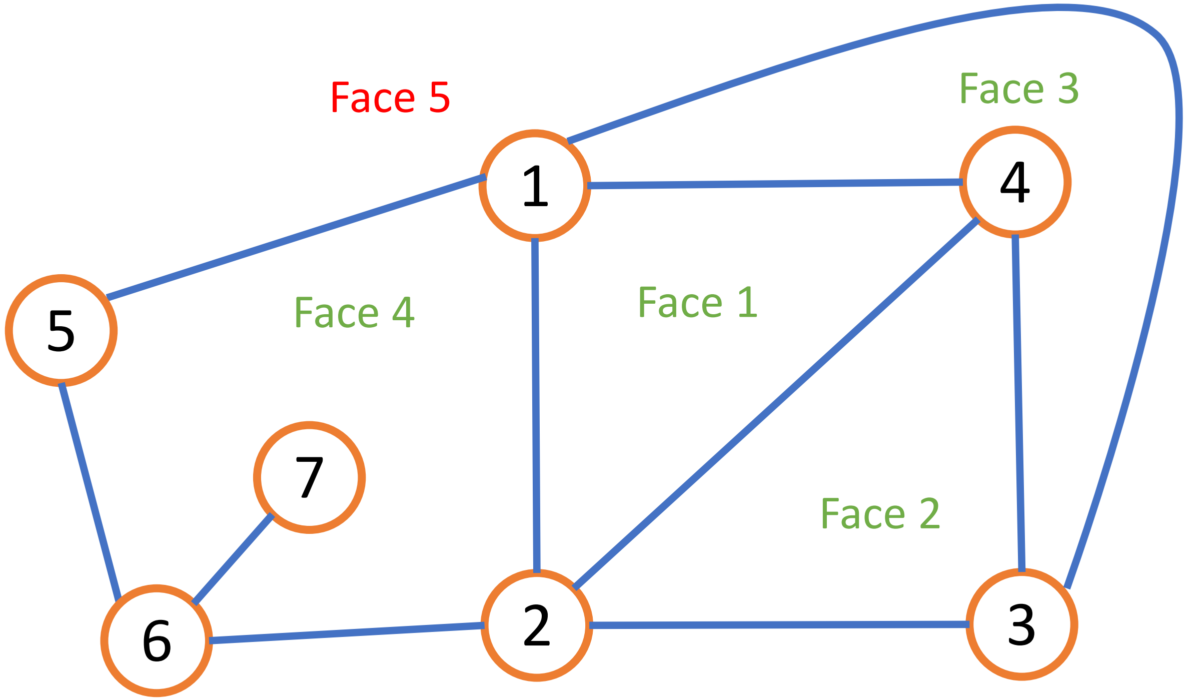


Dodecahedron

Euler's Formula

Theorem (Strengthened hypothesis)

A connected planar graph satisfies $v + f - e = 2$.



Proof of Euler's Formula

Proof.

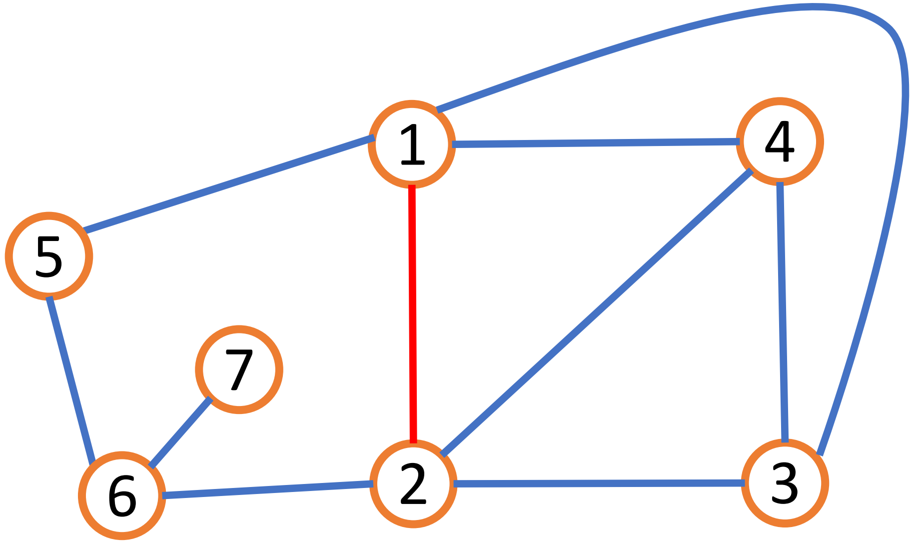
Base case: When $f = 1$, the graph is connected & acyclic => tree.

We have $e = v - 1$. $v + f - e = 2$

Induction hypothesis: Suppose the formula is tree for all graphs with f-1 faces.

Inductive Step: Take a graph with f faces.

We remove one edge separating two faces.



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Proof.

Base case: When $f = 1$, the graph is connected & acyclic => tree.

We have $e = v - 1$. $v + f - e = 2$

Induction hypothesis: Suppose the formula is tree for all graphs with f-1 faces.

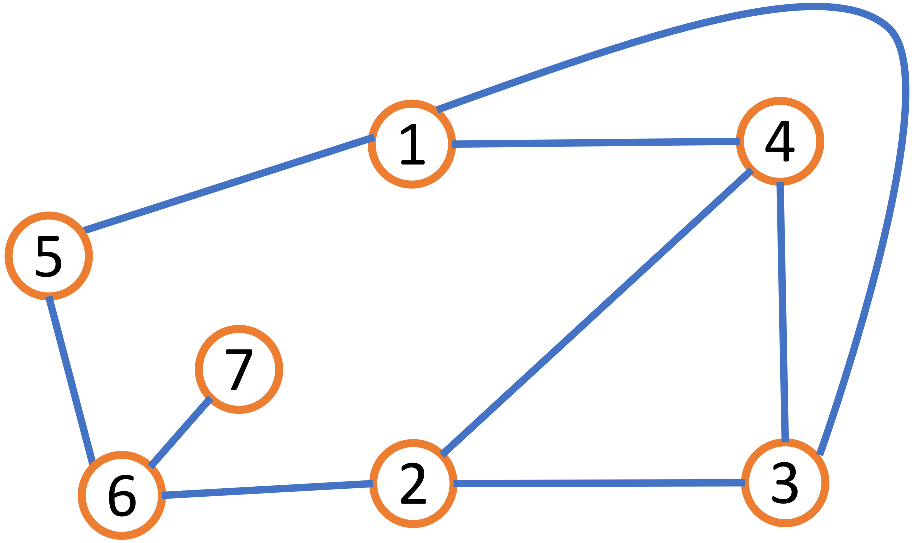
Inductive Step: Take a graph with f faces.

We remove one edge separating two faces.

f decrease by 1 and e decrease by 1.

We get a graph with $f - 1$ face.

………..



Proof of 3v-6 rule

Theorem

A connected planar graph can have at most $3v - 6$ edges.

Proof.

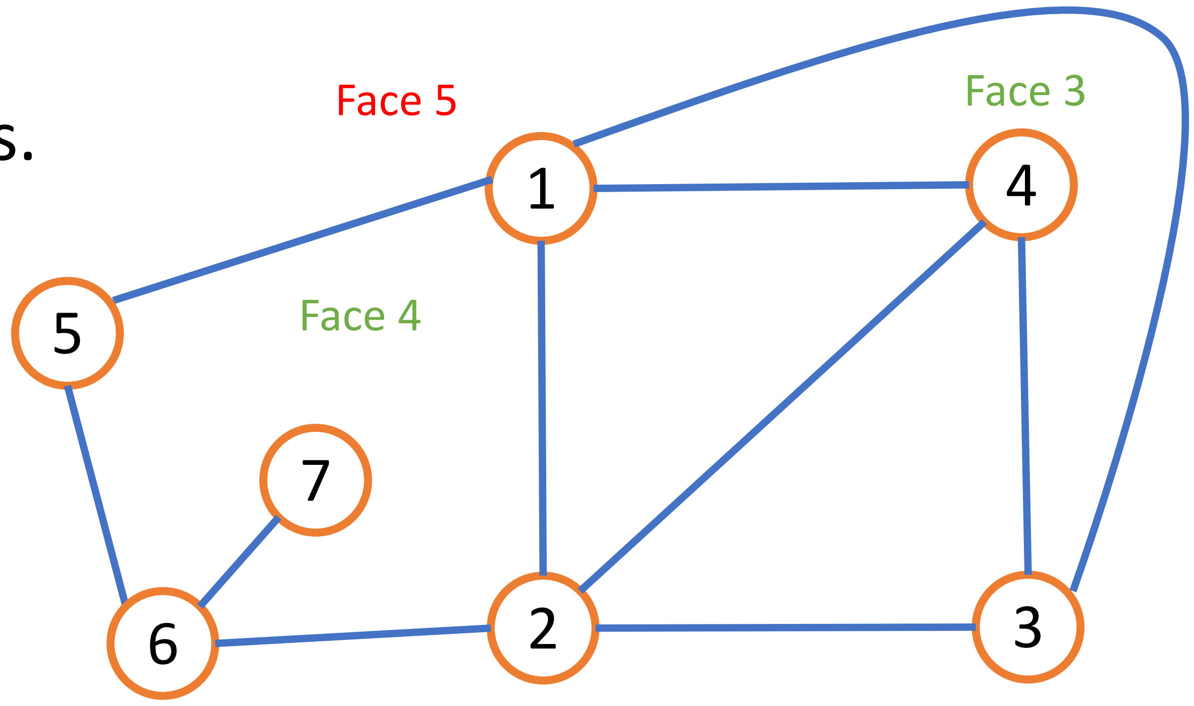
- A face is adjacent to ≥ 3 edges.
- An edge is adjacent to ≤ 2 faces.

We get $3f \leq 2e$.

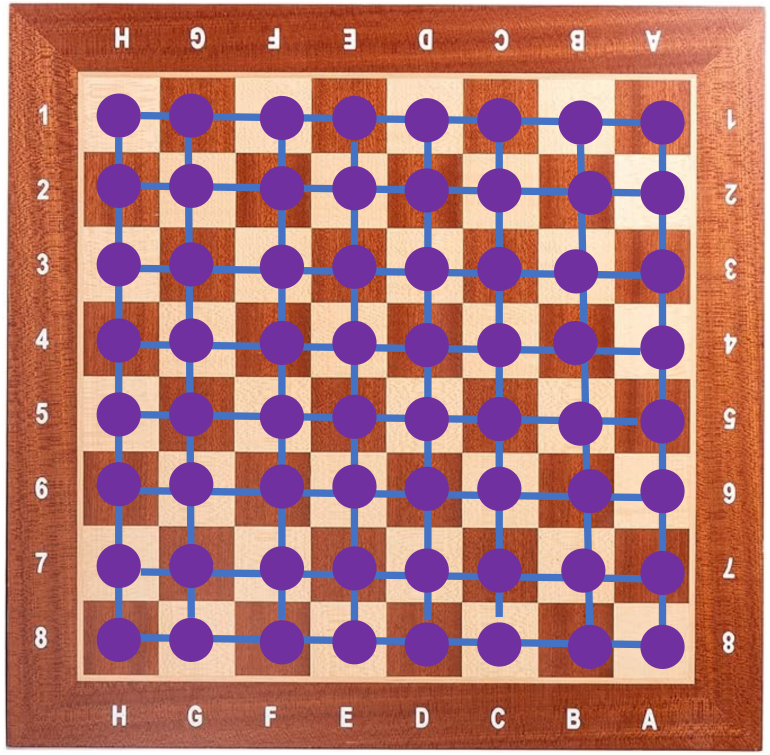
We know $v + f - e = 2$.

$f = e + 2 - v$,

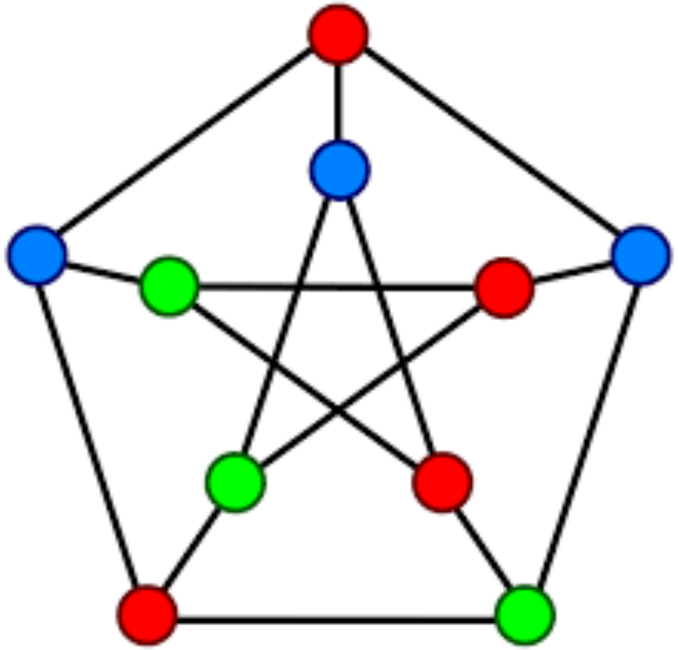
$3e + 6 - 3v \leq 2e$



Graph Coloring



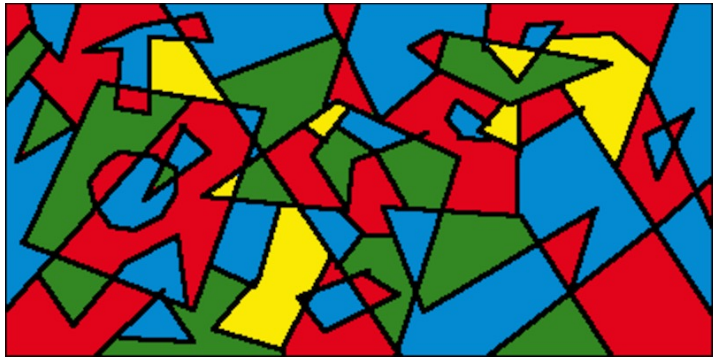
Two coloring

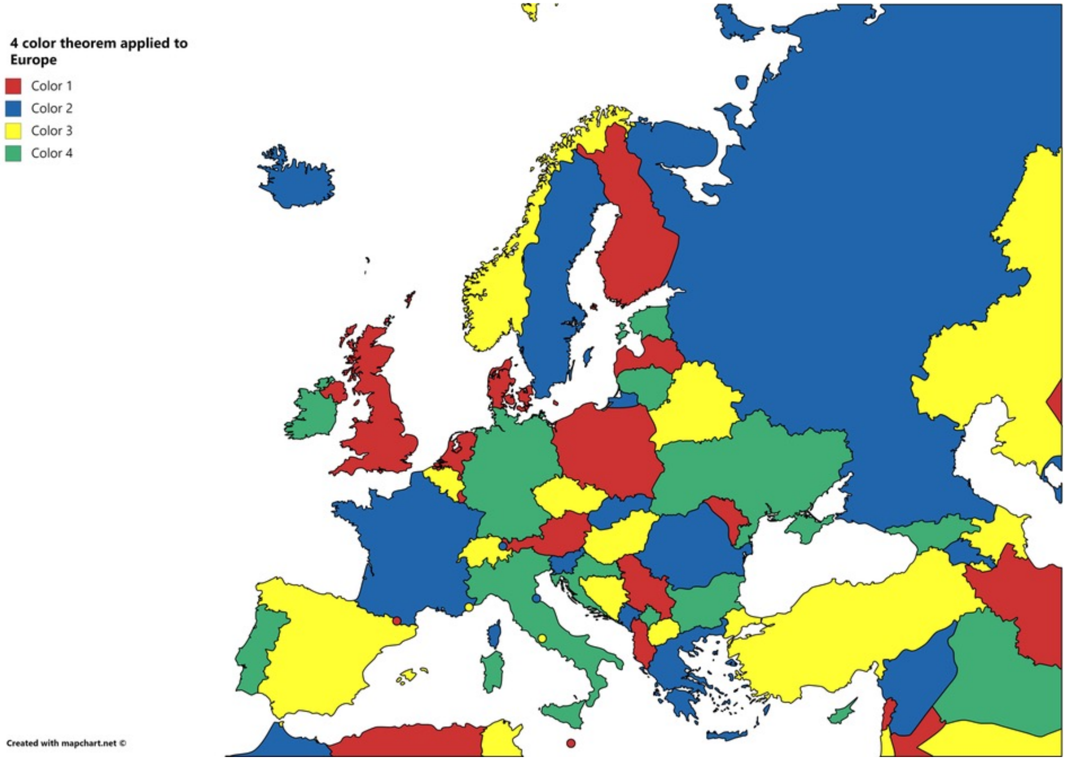


Three coloring

Four coloring theorem

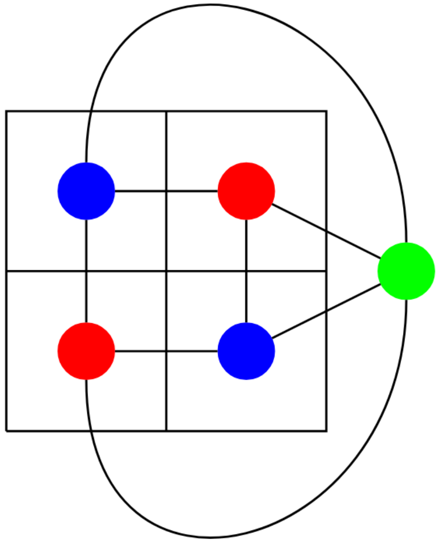
Theorem.
The regions on any map can be colored using four colors such that no adjacent regions have the same color.

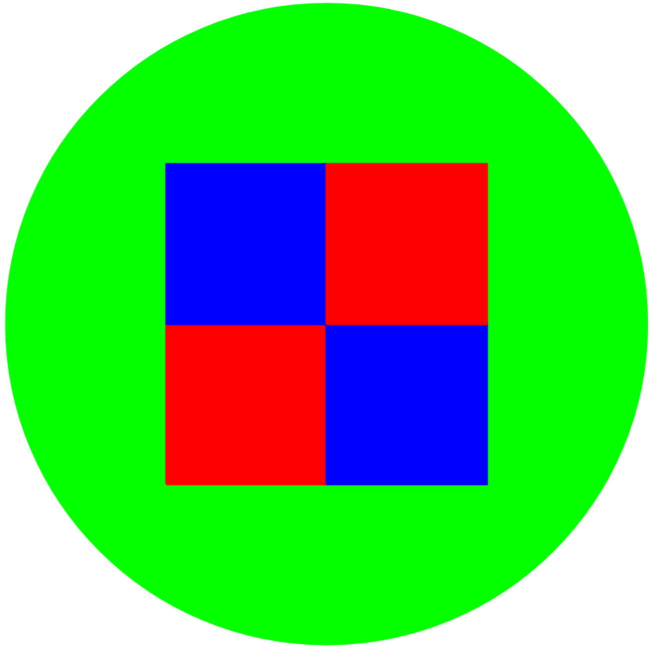




Four coloring theorem

Planar graph coloring $\equiv$ map coloring.





Degree + 1 Coloring

Theorem

It is always possible to color a graph with (maximum degree) + 1 colors.

.

Proof.

Simply color each vertex using a color that is different from all its neighbors.

(maximum degree) + 1 colors => never run out of color.

Six Coloring Theorem

Theorem

Any planar graph can be six-colored.

.

Proof.

$e \leq 3v - 6$ means average degree $\leq \frac{2\cdot(3v-6)}{v} < 6$.

So there exists a vertex with degree 5.

Remove that vertex, color the rest of the graph first (induction).

Add back that vertex, we never run out of color!

Hamiltonian Cycle

Theorem.

If a graph has minimum degree $\geq \frac{v}{2}$, then there is a Hamiltonian Cycle.

Proof.

Base Case: If v = 1, there is a Hamiltonian cycle.

Induction Hypothesis: Suppose this is true for v – 1.

Inductive Step: Take any graph with v vertices, remove an arbitrary vertex.

Hamiltonian Cycle

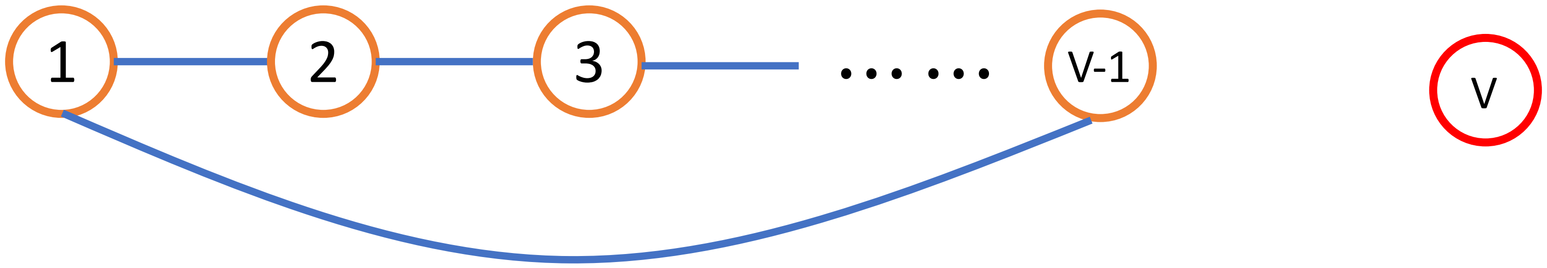
Theorem.

If a graph has minimum degree $\geq \frac{v}{2}$, then there is a Hamiltonian Cycle.

Proof.

Inductive Step: Take any graph with v vertices, remove an arbitrary vertex.

By induction hypothesis, the rest of the graph has a Hamiltonian Cycle.



Hamiltonian Cycle

Theorem.

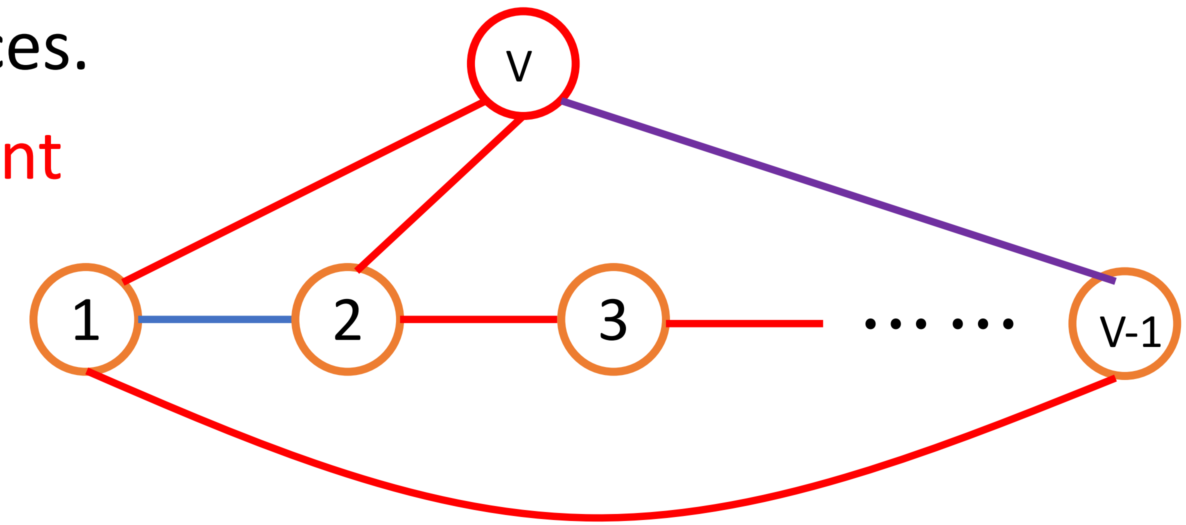
If a graph has minimum degree $\geq \frac{v}{2}$, then there is a Hamiltonian Cycle.

Proof.

Inductive Step: Removed vertex has degree $\frac{v}{2}$.

There are only v-1 previous vertices.

=> two neighbors must be adjacent



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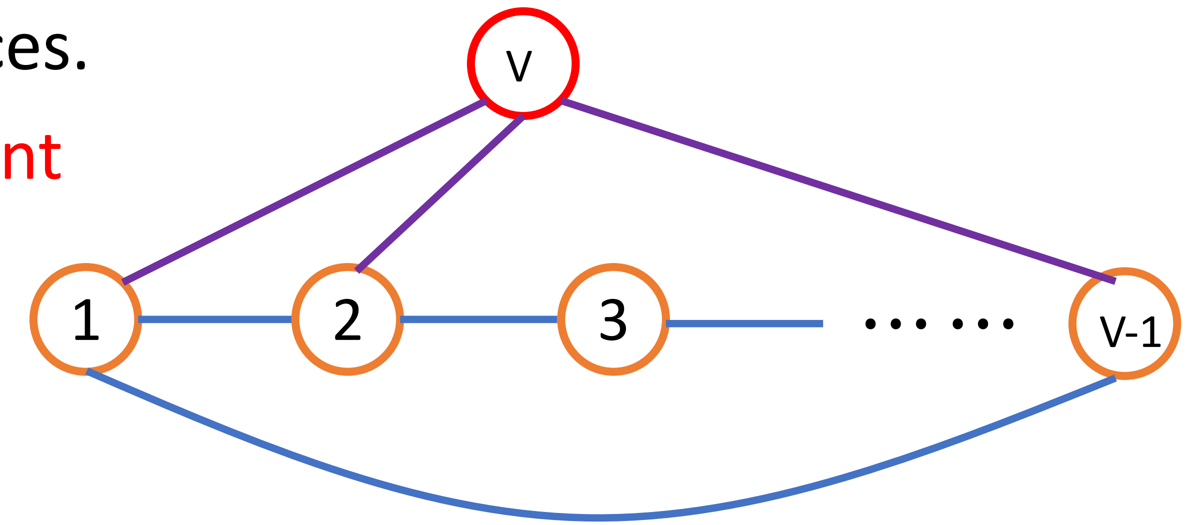
Splice!



Recent Advance

Hamiltonicity of expanders: optimal bounds and applications

Nemanja Draganić* Richard Montgomery † David Munhá Correia‡
Alexey Pokrovskiy§ Benny Sudakov‡

Abstract

An n-vertex graph G is a C-expander if $|N(X)| \geq C|X|$ for every $X \subseteq V(G)$ with $|X| < n/2C$ and there is an edge between every two disjoint sets of at least $n/2C$ vertices. We show that there is some constant $C > 0$ for which every C-expander is Hamiltonian. In particular, this implies the well known conjecture of Krivelevich and Sudakov from 2003 on Hamilton cycles in (n, d, λ)-graphs. This completes a long line of research on the Hamiltonicity of sparse graphs, and has many applications.

GRAPH THEORY

In Highly Connected Networks, There's Always a Loop

3

Mathematicians show that graphs of a certain common type must contain a route that visits each point exactly once.