Discussion 7C

CS 70, Summer 2024

1 Random Variables with Joint Density

Suppose that the random variables X and Y have the joint density

$$f(x, y) = \begin{cases} c(x-y)^2 & 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Find $P(X > 2Y)$.

(c) Find $P(X > Y + 1/2)$.

(d) Find the density of X.

(e) Determine whether X and Y are independent.

(f) Determine whether X and Y are identically distributed.

(g) For each $x \in (0, 1)$, find the conditional distribution Y given $X = x$.

(h) Find $P(Y \geq 1/3 \mid X = 1/2)$.

(i) Write down an integral expression for $E[XY]$. Your expression should only integrate densities over regions where they are nonzero.

2   The Erlang Distribution

For $\lambda > 0$, let $X_1, X_2, \ldots$ be independent and identically distributed exponential (λ) random variables. For each $r \geq 1$, let

$$T_r = X_1 + \ldots + X_r.$$

In this question, we will find the distribution of T_r.

(a) Find the distribution of T_1.

(b) Find the distribution of $T_2 = X_1 + X_2$.

(c) Prove that for any $r \geq 1$, T_r has density

$$f_r(t) = \frac{\lambda^r}{(r-1)!} t^{r-1} e^{-\lambda t}, \qquad t > 0.$$

This is known as the *Erlang* distribution. In the homework, you will find its survival function.

3 Competing Variables

(a) Let $\mu, \lambda > 0$. Suppose $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are independent. Find $P(X > Y)$.

(b) Suppose X and Y are independent and identically distributed with some density. Find $P(X > Y)$.

(c) Suppose X and Y are independent and identically distributed with some density. Find $P(|X| > |Y|)$.

(d) Suppose W, X, and Y are independent and identically distributed with some density. Find $P(W > X > Y)$.