Discussion 7A

CS 70, Summer 2024

1 Wet Professor

A professor has two umbrellas. Each morning, she walks from her apartment to her office, and each evening, she walks from her office to her apartment. It rains each morning and each evening with probability $p \in (0, 1)$, independent of all other mornings and evenings. If it is raining, the professor takes an umbrella if she has one with her; otherwise, she ignores the umbrellas.

If it so happens that neither of her umbrellas are with her and it is raining, she has no choice but to walk through the rain without an umbrella and get wet.

(a) Construct a Markov chain which represents this process. You should be able to use it to determine the long-run proportion of days where the professor gets wet.

(b) Prove that your Markov chain is irreducible and aperiodic.

(c) Find the long-run proportion of walks on which the professor gets wet.

(d) Suppose that the professor has no umbrellas with her. Find the expected number of walks until the professor gets wet.

2 Double Toss

Amari and Brynn are playing a game by repeatedly tossing a coin which lands heads with some probability $p \in [0, 1]$. If the coin lands heads twice in a row before it lands tails twice in a row, then Amari wins. Otherwise, Brynn wins.

For example, if the sequence of coin tosses is THTHH, then Amari wins the game.

(a) Find the chance that Amari wins the game.

(b) Find the expected number of tosses until the game ends.

(c) Find the expected number of heads until the game ends.

3   Die Sum

Ngozi repeatedly rolls a six-sided die with faces numbered 1 through 6 and sums the rolls. Let S_n be Ngozi's sum modulo 3 after n rolls.

(a) Construct a Markov chain which represents this process.

(b) Prove that your Markov chain is irreducible and aperiodic.

(c) Find the stationary distribution of the Markov chain.

(d) Suppose Ngozi's current sum is a multiple of 3. Find the expected number of rolls until her sum is once again a multiple of 3.