Discussion 6A

1 Balls and Bins

Throw n balls into m bins, where m and n are positive integers. Each ball i lands independently in a uniformly random bin. Let X be the number of bins with exactly one ball.

(a) Let X_i be the indicator that bin i has exactly one ball, for each $i = 1, \ldots, m$. What is $\Pr[X_i = 1]$?

(b) What is $\mathrm{E}[X]$?

(c) For $i \neq j$, what is $\mathrm{E}[X_i X_j]$? How about when $i = j$?

(d) Compute $\mathrm{Var}(X)$. Your final answer should not contain any summations.

2 Correlation and Independence

(a) What does it mean for two random variables X, Y to be uncorrelated?

(b) What does it mean for two random variables X, Y to be independent?

(c) Are all uncorrelated variables independent? Are all independent variables uncorrelated? If your answer is yes, justify your answer; if your answer is no, give a counterexample.

3   Diverse Hand

You dealt 5 cards (wihout replacement) from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(There 13 different values in a 52 card deck and each value is on four cards.)

  (a) Calculate $E[X]$. (Hint: Consider indicator variables X_i representing whether i appears in the hand.)

  (b) Calculate $\text{Var}(X)$.

4   Covariance

  (a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\text{Cov}(X_1, X_2)$? Recall that $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$.

(b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\mathrm{Cov}(X_1, X_2)$?