Discussion 1B

CS 70, Summer 2024

1 Rationals and Irrationals

Consider the proposition "the product of a nonzero rational number and an irrational number is irrational."

(a) For a real number x, let $Q(x)$ be the predicate "x is rational" and let $I(x)$ be the predicate "x is irrational." Express $Q(x)$ and $I(x)$ in first-order logic without any leading negations and without using $\mathbb{Q}$.

(b) Use the predicates Q and I to state the proposition in first-order logic.

(c) Prove the proposition.

2 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue.

- Let P be the proposition "there exists a column of pebbles that are all red."
- Let Q be the proposition "for every way of choosing one pebble from each column, there exists a red pebble among the chosen pebbles".

(a) Prove that $P \implies Q$.

(b) Prove that $Q \implies P$.

3 Numbers of Friends

Suppose there are $n \geq 2$ people at a party, some of which are friends with one another. Suppose that friendships are reciprocated: that is, if Aditya is friends with Belinda, then Belinda is also friends with Aditya.

(a) Find the range of the possible number of friends a partgoer can have at the party.

(b) If Aditya is at the party and has no friends at the party, can there be anybody at the party with $n-1$ friends? Explain.

(c) The pigeonhole principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item.

Use the pigeonhole principle to prove that if there are $n \geq 2$ people at the party, then at least two of them will have the same number of friends.

4 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p must be of the form $3k + 1$ or $3k - 1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part **(a)** to prove that 5 is the only prime number that takes part in two different twin prime pairs.